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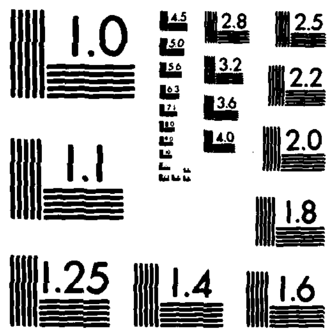

MICROCOPY RESOLUTION TEST CHART

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OPERATIONS RESEARCH CENTER

RANDOM SEARCH FOR A PROBABLE OBJECT

by

William S. Jewell†

ORC 85-5 June 1985

UNIVERSITY OF CALIFORNIA

BERKELEY

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ABSTRACT

A search is undertaken for an object thought to be present with an unknown probability, using a detection scheme whose efficiency is also uncertain. After a certain interval, the search is called off, with the object unfound; what are the posterior-to-experiment estimates of presence and detection efficiency? It is shown that those two unknown quantities interact in an interesting manner as the unsuccessful search goes on.

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A RANDOM SEARCH FOR A PROBABLE OBJECT

by

William S. Jewell

0. INTRODUCTION

Suppose that an object with certain distinct characteristics is thought to be present in a given region or among a large group of similar objects. At time zero, a search for this object begins, using equipment or procedures whose detection efficiency is not known precisely. Then, after a certain interval of time t, the search is called off, with the object unfound; shall we say that the object was undiscovered because it was not actually there in the first place, because the search time was inadequate, or because the search process was not efficient enough?

This type of problem occurs in software debugging, where an observed error might be thought to be due to a bug in a particular portion of a program, when in fact it is due to the operating system, the hardware, line fluctuations, etc. Another similar situation occurs in quality control of mass-produced parts, where the testing is done in batches; a test could reveal that a defective part might exist in the batch, but in fact the procedure may not be precise enough to reveal just one defect. Another application, of great topical interest, has to do with the recent incidents of intrusion of territorial waters by foreign submarines.

Because this problem concerns the change between initial and post-experimental perceptions of certain parameters, we use a Bayesian formulation, whose analysis is straightforward, even under general assumptions about the priors. However, the results are somewhat counter-intuitive, particularly when considered as functions of t, because of the interactions between the different sources of uncertainty. This may account for some of the extended discussion following the international incidents just mentioned.

1. BASIC MODEL

For the basic model, we assume that, if the object exists at time zero (event E) in the given region, then the random time until detection, $\tilde{\tau}$, is given by an exponential law with parameter ϕ. This is consistent with various randomized search procedures in which the probability that an existing object, not found by time t, will be discovered in the next dt is $(\phi\, dt)$, independent of past history; it follows that:

$$\Pr\{\tilde{\tau} > t \mid E, \phi\} = e^{-\phi t} \qquad (t \geq 0) \tag{1}$$

Note that ϕ may be a characteristic of the target, the region of search, the search procedures, the detection threshold of the equipment-observer combination, or all of these. Thus, by an uncertain search efficiency, we mean that the discovery rate is treated as a random quantity, $\tilde{\phi}$, with known prior density, $p(\phi)$.

Let π be the probability that the object is actually present at time zero, so that $\Pr\{\bar{E}\} = 1 - \pi$; again, our problematic attitude about E can be reflected by making $\tilde{\pi}$ a random quantity, with known prior density. $p(\pi)$ is determined by collateral measurements of the object's presence, "inside information," expert opinion, external related events, and "other signs and symptoms." It seems reasonable in most applications to make $\tilde{\phi}$ and $\tilde{\pi}$ independent, a priori.

If the search is called off at epoch t with the putative object unfound, then we have an experiment in which the "data" is $\mathcal{D}_t = [\tilde{\tau} > t]$, and it follows that the likelihood is simply:

$$\Pr\{\mathcal{D}_t \mid \pi, \phi\} = (1 - \pi) + \pi e^{-\phi t} \quad . \qquad (t \geq 0) \tag{2}$$

Then, a simple application of Bayes' law will determine the posterior joint density of the unknown parameters, $p(\pi, \phi \mid \mathcal{D}_t)$, with other quantities of interest following from that. However, our final result can be better interpreted if we first consider two related elementary experiments.

2. ELEMENTARY EXPERIMENTS

Consider first a search in which the detection efficiency is very high, or which has been in progress for a very long time. Thus, we know "for sure" either that the object is present or not, that is, the datum is either E or $\bar{E}$; this amounts to a Bernoulli experiment with one trial. Then, under similar circumstances in the future, we would use a different prior on $\tilde{\pi}$, based upon the outcome of *this* trial. In other words, our prior would change to either of:

$$p(\pi \mid \bar{E}) = \frac{(1 - \pi)\, p(\pi)}{1 - E\{\tilde{\pi}\}} \quad ; \quad p(\pi \mid E) = \frac{\pi\, p(\pi)}{E\{\tilde{\pi}\}} \quad . \tag{3}$$

Physically speaking, the outcome of this experiment reflects upon our intelligence-gathering procedures, preliminary testing, or other means of estimating $\tilde{\pi}$.

We can get some idea of the shift of opinion by examining the posterior means:

$$E\{\tilde{\pi} \mid \bar{E}\} = E\{\tilde{\pi}\} - V\{\tilde{\pi}\}/[1 - E\{\tilde{\pi}\}] \ ; \ E\{\tilde{\pi} \mid E\} = E\{\tilde{\pi}\} + V\{\tilde{\pi}\}/E\{\tilde{\pi}\}$$

which shows that the amount of shift down $(\bar{E})$ or up (E) is related to the prior variance, $V\{\tilde{\pi}\}$. In particular, if our prior opinion can be modelled by a Beta (α,β) density, then we know the posterior density is closed under sampling, with $p(\pi \mid \bar{E})$ being Beta $(\alpha,\beta+1)$, and $p(\pi \mid E)$ being Beta $(\alpha+1,\beta)$. The posterior means are then:

$$E\{\tilde{\pi} \mid \bar{E}\} = (\delta\, E\{\tilde{\pi}\} + 1)/(\delta + 1) \quad ; \quad E\{\tilde{\pi} \mid E\} = \delta\, E\{\tilde{\pi}\}/(\delta + 1)$$

where $E\{\tilde{\pi}\} = \alpha/\delta$, and $\delta = \alpha + \beta$ represents the "strength" of prior opinion (essentially the peakedness of the mode of $p(\pi)$), relative to the experimental outcome, in limiting shifting of the mean.

A contrasting simple experiment is one in which we know the object is present, but once again search during an interval $(0, t]$ and do not find the object. The data from this experiment, $\mathcal{D}_t^* = \{(\tilde{\tau} > t) \cap E\}$, thus modifies only our prior opinion about search efficiency, giving an updated posterior density:

$$p(\phi \mid \mathcal{D}_t^*) = e^{-\phi t}\, p(\phi)/f(t) \;, \tag{4}$$

where the normalization

$$f(t) = E\left\{e^{-\tilde{\phi} t}\right\} = \int e^{-\phi t}\, p(\phi)\, d\phi \tag{5}$$

is the transform of $p(\phi)$. Thus this "failed" experiment shifts the mass and moments of $p(\phi)$ towards zero. In fact, it is easy to show that the partial ordering

$$\tilde{\phi} \gtrsim \tilde{\phi} \mid \mathcal{D}_s^* \gtrsim \phi \mid \mathcal{D}_t^* \tag{6}$$

holds for all $0 \le s \le t$.

Of particular interest is the behavior of mean posterior efficiency, gotten by using asymptotic forms for $f(t)$ and the convenient formula

$$E\{\tilde{\phi} \mid \mathcal{D}_t^*\} = -\frac{d \ln f(t)}{dt} \;.$$

For example, if t is small:

$$f(t) \approx 1 - E\{\tilde{\phi}\}\, t + E\{\tilde{\phi}^2\}\, t^2/2 - \ldots + \ldots \quad , \tag{7}$$

assuming the indicated moments exist, we find that:

$$E\{\tilde{\phi} \mid \mathcal{D}_t^*\} \approx E\{\tilde{\phi}\} - V\{\tilde{\phi}\}\, t + \ldots \qquad (t \to 0) \; . \tag{8}$$

For large t , assume that $p(\phi)$ has an analytic expansion at the origin:

$$p(\phi) = \sum_{j=J}^{\infty} p_j\, \phi^j/j!$$

with possibly the first few coefficients vanishing, say $p_0 = p_1 = \ldots\, p_{J-1} = 0$. Then we have

$$f(t) \approx p_J\, t^{-(J+1)} + p_{J+1}\, t^{-(J+2)} + \ldots \, , \qquad (t \to \infty) \tag{9}$$

so that the posterior mean vanishes with t as

$$E\{\tilde{\phi} \mid \mathcal{D}_t^*\} \approx (J+1)t^{-1} + (p_{J+1}/p_J)t^{-2} + \ldots \; . \qquad (t \to \infty) \tag{10}$$

The reason we assume that the first non-zero coefficient might be $J > 0$ has to do with the moments of $\tilde{\phi}^{-1}$, which are of independent interest; for example, the posterior mean of $\tilde{\phi}^{-1}$ is the expected *remaining* time until discovery (see Section 3). Assuming $E\{\tilde{\phi}^{-1}\}$ is finite, we find, similar to (8) above:

$$E\{\tilde{\phi}^{-1} \mid \mathcal{D}_t^*\} \approx E\{\tilde{\phi}^{-1}\} + \left[E\{\tilde{\phi}\}E\{\tilde{\phi}^{-1}\} - 1\right]t - \ldots + \ldots \, , \quad (t \to 0) \tag{11}$$

and, assuming $J > 1$:

$$E\{\tilde{\phi}^{-1} \mid \mathcal{D}_t^*\} \approx \frac{t}{J} - \left(\frac{p_{J+1}}{p_J}\right)\frac{1}{J(J+1)} + \ldots \; . \qquad (t \to \infty) \tag{12}$$

For more detailed information, we need to use a specific prior; a convenient choice is the closed-under-sampling Gamma(c, d) density, for which $f(t) = [1 + (t/d)]^{-c}$, and $p(\phi \mid \mathcal{D}_t^*)$ is simply Gamma(c, d + t). Then, for all $t \geq 0$:

$$E\{\tilde{\phi} \mid \mathcal{D}_t^*\} = \frac{c}{d + t} = \frac{E\{\tilde{\phi}\}}{1 + (t/d)} \quad , \tag{13}$$

and, assuming $c > 1$:

$$E\{\tilde{\phi}^{-1} \mid \mathcal{D}_t^*\} = \frac{d + t}{c - 1} = E\{\tilde{\phi}^{-1}\} + \frac{t}{c - 1} \, , \tag{14}$$

In words, we find for this simple experiment $\mathcal{D}_t^*$ that, where the object is known to be present, our revised estimate of the mean rate of discovery is dropping towards zero with increasing duration of non-discovery, or, equivalently, that the mean remaining time until discovery is increasing without limit with elapsed time, in the limit as a linear function of t. Incidentally, this example, which holds for arbitrary $c > 1$, shows that the assumption of an analytic expansion about the origin is too strong, and that J can be non-integer as well.

3. INTERPRETATION OF THE COMPOUND EXPERIMENT

Returning to the experiment in which both $\tilde{\pi}$ and $\tilde{\phi}$ are uncertain, we use (2) and the notation of the last Section to find the joint posterior density:

$$p(\pi, \phi \mid \mathcal{D}_t) = [1 - k(t)] \; p(\pi \mid \bar{E}) \; p(\phi) + k(t) \; p(\pi \mid E) \; p(\phi \mid \mathcal{D}_t^*) \; , \tag{15}$$

which is seen to be a mixture of the two elementary experiments just analyzed. The time-dependent mixing coefficient:

$$k(t) = \frac{E\{\tilde{\pi}\}f(t)}{1 - E\{\tilde{\pi}\} + E\{\tilde{\pi}\}f(t)} = \frac{\omega f(t)}{1 + \omega f(t)} \quad , \tag{16}$$

depends upon $p(\pi)$ through the prior expected odds, $\omega = E\{\tilde{\pi}\}/[1 - E\{\tilde{\pi}\}]$, and upon $p(\phi)$ through $f(t) = E\{\exp(-\tilde{\phi}t)\}$. It is not difficult to show that $k(t)$ is monotone decreasing from $E\{\tilde{\pi}\}$ to zero as t goes from zero to infinity; note that the first component in (15) never vanishes, nor does the second component ever receive full weight.

Under the conditions described in the last Section, we find the limiting approximations:

$$k(t) \approx E\{\tilde{\pi}\} - E\{\tilde{\pi}\}\ [1 - E\{\tilde{\pi}\}]\ E\{\tilde{\phi}\}\ t + \ldots \ , \qquad (t \to 0) \tag{17}$$

and

$$k(t) \approx p_J\ t^{-(J+1)} + p_{J+1}\ t^{-(J+2)} + \ldots \quad , \qquad (t \to \infty) \tag{18}$$

where we henceforth assume $J \geq 1$.

Specifically, if $p(\phi)$ is Gamma(c, d), we find

$$k(t) = \omega/[\omega + (1 + (t/d))^c] \ .$$

Figure 1 shows $k(t)$ and $1 - k(t)$ when $E\{\tilde{\pi}\} = 0.75$ $(\omega = 3{:}1)$, and $p(\phi)$ is Gamma(6,50), so that $E\{\tilde{\phi}\} = 0.12$ and $E\{\tilde{\phi}^{-1}\} = 10.0$.

The effect of the model-mixing in (15) can be seen more clearly by examining the posterior marginal densities. First, since

$$p(\pi \mid \mathcal{D}_t) = [1 - k(t)]\ p(\pi \mid \bar{E}) + k(t)\ p(\pi \mid E) \ ,$$

we find that the posterior-to-$\mathcal{D}_t$ expectation of the probability that the object was actually present is:

$$E\{\tilde{\pi} \mid \mathcal{D}_t\} = \frac{E\{\tilde{\pi}\} - E\{\tilde{\pi}^2\}\,[1 - f(t)]}{1 - E\{\tilde{\pi}\}\,[1 - f(t)]} \quad . \tag{19}$$

It follows easily that this estimator decreases monotonically with t from its prior expectation, $E\{\tilde{\pi}\}$, to the value $E\{\tilde{\pi} \mid \bar{E}\}$. Limiting forms are:

$$E\{\tilde{\pi} \mid \mathcal{D}_t\} \approx \begin{cases} E\{\tilde{\pi}\} - V\{\tilde{\pi}\}\,E\{\tilde{\phi}\}\,t + \ldots - \ldots & (t \to 0) \\ E\{\tilde{\pi} \mid \bar{E}\} + \dfrac{V\{\tilde{\pi}\}}{[1 - E\{\tilde{\pi}\}]^2}\,p_J\,t^{-(J+1)} + \ldots & (t \to \infty) \end{cases} \tag{20}$$

Figure 2 shows $E\{\tilde{\pi} \mid \mathcal{D}_t\}$ for the Gamma-based $f(t)$ used in Figure 1, and for $E\{\tilde{\pi}\} = 0.75$ and $E\{\tilde{\pi} \mid \bar{E}\} = 0.60$, obtained from $p(\pi)$ being Beta(3,1).

The interesting part of this result is not the monotonicity with t, which is expected, but is the fact that the total decrease is so small in most cases; in fact, only the fraction $V\{\tilde{\pi}\}/E\{\tilde{\pi}\}\,[1 - E\{\tilde{\pi}\}]$ of the original estimate, $E\{\tilde{\pi}\}$, is lost when the search is called off after a very long time t. If $p(\pi)$ is Gamma, this fraction is only $(\delta + 1)^{-1}$ of the original value; unless we make the extreme assumption that $p(\pi)$ is "dumbbell"-shaped, δ must be larger than 2. The reason why (19) does not vanish in the limit is discussed below in Section 5.

Turning now to the posterior behavior of $\tilde{\phi}$, since

$$p(\phi \mid \mathcal{D}_t) = [1 - k(t)]p(\phi) + k(t)\,p(\phi \mid \mathcal{D}_t^*) \ ,$$

we have a posterior mean efficiency of

$$E\{\tilde{\phi} \mid \mathcal{D}_t\} = E\{\tilde{\phi}\} - \frac{\omega\left[E\{\tilde{\phi}\}\,f(t) + \dfrac{df(t)}{dt}\right]}{[1 + \omega\,f(t)]} \tag{21}$$

From the expansions of the last Section, we find:

$$E\{\tilde{\phi} \mid \mathcal{D}_t\} \approx \begin{cases} E\{\tilde{\phi}\} - E\{\tilde{\pi}\}\, V\{\tilde{\phi}\}\, t + \ldots & (t \to 0) \\ E\{\tilde{\phi}\} - \omega E\{\tilde{\phi}\}\, p_J\, t^{-(J+1)} + \ldots & (t \to \infty) \end{cases} . \tag{22}$$

Thus we obtain the surprising result that, for most reasonable $p(\phi)$, the posterior mean $E\{\tilde{\phi} \mid \mathcal{D}_t\}$ approaches the prior mean, $E\{\tilde{\phi}\}$, for <u>both t small and large</u>, usually with a unique <u>minimal value</u> in between!

For instance, with $p(\phi)$ being Gamma(c, d), one can find that

$$E\{\tilde{\phi} \mid \mathcal{D}_t\} = E\{\tilde{\phi}\} \left[\frac{1 + \omega[1 + (t/d)]^{-(c+1)}}{1 + \omega[1 + (t/d)]^{-c}} \right] ,$$

which is shown in Figure 3, using the same Gamma(6,50) used in Figure 1. The minimal value of the posterior mean, which is down about 8.9% from the initial and asymptotic value of $E\{\tilde{\phi}\} = 0.12$, occurs at about $t = 14$. Also shown on the same plot is the corresponding plot of $E\{\tilde{\phi} \mid \mathcal{D}_t^*\} = c/(d+t)$ from (13).

The physical explanation for the shape of (21) is that, for small t, we at first think that non-detection implies that the detection rate should be reduced, as in the simpler case with data $\mathcal{D}_t^*$. However, at the same time, we are also reducing our expectation on $\tilde{\pi}$, which puts more weight on the hypothesis $\bar{E}$; under this hypothesis, there is <u>no</u> information about $\tilde{\phi}$ to be obtained from the experiment! Therefore, as t gets very large, the first term in the marginal posterior dominates, and the posterior mean returns to $E\{\tilde{\phi}\}$.

4. TOTAL DISCOVERY RATE

The total discovery rate may be defined as $\tilde{\rho} = \tilde{\pi}\,\tilde{\phi}$, that is, (ρdt) is the probability that the object is found in $(t, t + dt)$, conditional only on $\mathcal{D}_t$. This concept is useful in making marginal decisions about whether or not to continue the search, given the relative costs of continuing to look and the value of finding the object. For example, if search costs are c \$/unit time, and the value of finding the object is V \$, both constant over time, then the strategy which maximizes the expected net value is to continue the search as long as $E\{\tilde{\rho} \mid \mathcal{D}_t\} > c/V$, if not yet found.

From previous formulae we find the mean discovery rate to be the cross-moment of (15):

$$E\{\tilde{\rho} \mid \mathcal{D}_t\} = \frac{E\{\tilde{\pi}\}\, E\{\tilde{\phi}\} - E\{\tilde{\pi}^2\}\, [E\{\tilde{\phi}\} - E\{\tilde{\phi}\, e^{-\tilde{\phi} t}\}]}{1 + E\{\pi\}\, [1 - f(t)]} \quad . \tag{23}$$

The asymptotic values are obtained in the same way as before, giving:

$$E\{\tilde{\rho} \mid \mathcal{D}_t\} \approx \begin{cases} E\{\tilde{\pi}\}\, E\{\tilde{\phi}\} - \left[E\{\tilde{\pi}^2\}\, E\{\tilde{\phi}^2\} - [E\{\tilde{\pi}\}\, E\{\tilde{\phi}\}]^2\right] t + \ldots & (t \to 0) \\ E\{\tilde{\pi} \mid \bar{E}\}\, \left[E\{\tilde{\phi}\} - \omega\, p_J\, t^{-(J+1)} + \ldots\right] & (t \to \infty) \end{cases} \tag{24}$$

Both asymptotic value are expected, but what is interesting here is that the average total discovery rate is <u>not</u> monotonic, but undershoots its final value!

This phenomenon is shown in Figure 4, using the same Beta-Gamma parameters as in previous examples; this gives an undershoot of about 2.5% at about $t = 30$. The cause of this phenomenon is the "recovery" of $E\{\tilde{\phi} \mid \mathcal{D}_t\}$ with t , since, if we knew $\phi = \phi_0$ "for sure," then $E\{\tilde{\rho} \mid \mathcal{D}_t\}$ would be $\phi_0\, E\{\tilde{\pi} \mid \mathcal{D}_t\}$, which we have already shown is monotone.

5. CERTAIN VERSUS UNCERTAIN DISCOVERY

Many people who are willing to accept the somewhat counter-intuitive behavior of $E\{\tilde{\phi} \mid \mathcal{D}_t\}$ and $E\{\tilde{\rho} \mid \mathcal{D}_t\}$ versus t are still bothered by the non-vanishing of $E\{\tilde{\pi} \mid \mathcal{D}_t\}$; why don't we conclude that the object was not present after a very long time of search? This is essentially related to the conditions implied by $\mathcal{D}_t$ and by $\mathcal{D}_t^*$, and the related behavior of $\tilde{\phi}$. If, in fact, the object were present (and we did not know it), then the probability of actually observing $\mathcal{D}_t$ (i.e., not finding the object), with t very large, would be vanishingly small. Nevertheless, if $\mathcal{D}_t$ ever occurs, we must, in this idealized setup, always hedge our bets on this search (and on all similar future searches). So we are not permitted to say the object was <u>for certain</u> not present as t gets very large.

On the other hand, if, at some point during this experiment, supplementary information on E is obtained, then of course we must revise the current probabilities depending upon the credulity of the new knowledge. In particular, if at time t we now know <u>for certain</u> that the object is present, then $\tilde{\pi}$ becomes unity "for sure," and we would switch to $p(\phi \mid \mathcal{D}_t^*)$ for the detection rate. One can easily show that

$$\tilde{\phi} \gtrsim \tilde{\phi} \mid \mathcal{D}_t \gtrsim \tilde{\phi} \mid \mathcal{D}_t^* \tag{25}$$

for all t , so that there is a discontinuity on our estimate of $\tilde{\phi}$ when this new knowledge is acquired.

If E becomes certain at t , it is also of interest to inquire what the distribution of <u>remaining time until discovery</u> will be, call it $\tilde{\tau}_t = \tilde{\tau} - t$. Now,

$$\Pr\left\{\tilde{\tau}_t > u \mid \mathcal{D}_t^* , \phi\right\} = e^{-\phi u}$$

for all t, according to (1). On the other hand, $p(\phi \mid \mathcal{D}_t^*)$ does depend upon t through the equation before (21), so we find the survival law:

$$\Pr\left\{\tilde{\tau}_t > u \mid \mathcal{D}_t^*\right\} = E\left\{e^{-\tilde{\phi}u} \mid \mathcal{D}_t^*\right\} = \frac{f(t+u)}{f(t)} \quad . \tag{26}$$

This gives a new interpretation to the transform f as the tail distribution function of $\tilde{\tau}_0 = \tilde{\tau}$.

The fact that a transform is log-convex also gives the interesting result that $\tilde{\tau} \lesssim \tilde{\tau}_s \lesssim \tilde{\tau}_t$ for all $0 \leq s \leq t$. Or, in words, even though it is known that the object is present, the longer the object remains unfound, the longer remaining time, on the average, it will take to find it (assuming $p(\phi)$ is not concentrated at a single point). A most discouraging outcome!

For example, if $p(\phi)$ is Gamma(c, d), then

$$\Pr\left\{\tilde{\tau}_t > u \mid \mathcal{D}_t^*\right\} = \left[\frac{1 + (t/d)}{1 + ((t+u)/d)}\right]^c \quad .$$

Of course, $E\{\tilde{\tau}_t \mid \mathcal{D}_t^*\} = E\{\tilde{\phi}^{-1} \mid \mathcal{D}_t^*\}$, as given in (14).

One could also define a remaining time until detection in the more general case, where either E or $\bar{E}$ may obtain, by writing

$$\Pr\left\{\tilde{\tau}_t > u \mid \mathcal{D}_t, \phi\right\} = \tilde{\pi}\, e^{-\tilde{\phi}u} + (1 - \tilde{\pi}) \quad ,$$

giving, formally:

$$\Pr\left\{\tilde{\tau}_t > u \mid \mathcal{D}_t\right\} = E\left\{\tilde{\pi} \mid \mathcal{D}_t\right\} + \frac{E\{\tilde{\pi} \mid \bar{E}\}\,[f(u) - f(t+u)] + f(t+u)}{1 + \omega f(t)} \quad .$$

The first term is the defect, reflecting the mass at $\tau_t^* = \infty$.

6. CONCLUDING REMARKS

This model can be generalized in several different ways. For example, in certain kinds of intermittent electronic failure problems, the bug is actually "present" or "not present" for alternating intervals that depend upon non-observable externalities; the prior would require hypothesizing where in this cycle the search begins. Similar models arise if the detection efficiency varies randomly between several different levels, as in various communications problems. In intrusion detection, the target may actually be able to detect the search effort and take evasive action, or leave the region; this requires modelling the time until the object has left, given it was originally present.

In summary, we have found that the original uncertainties about an object's presence and our detection abilities interact in an interesting way as time elapses with the object unfound. In particular, posterior estimates of these uncertainties may be non-monotonic with the non-detection interval.

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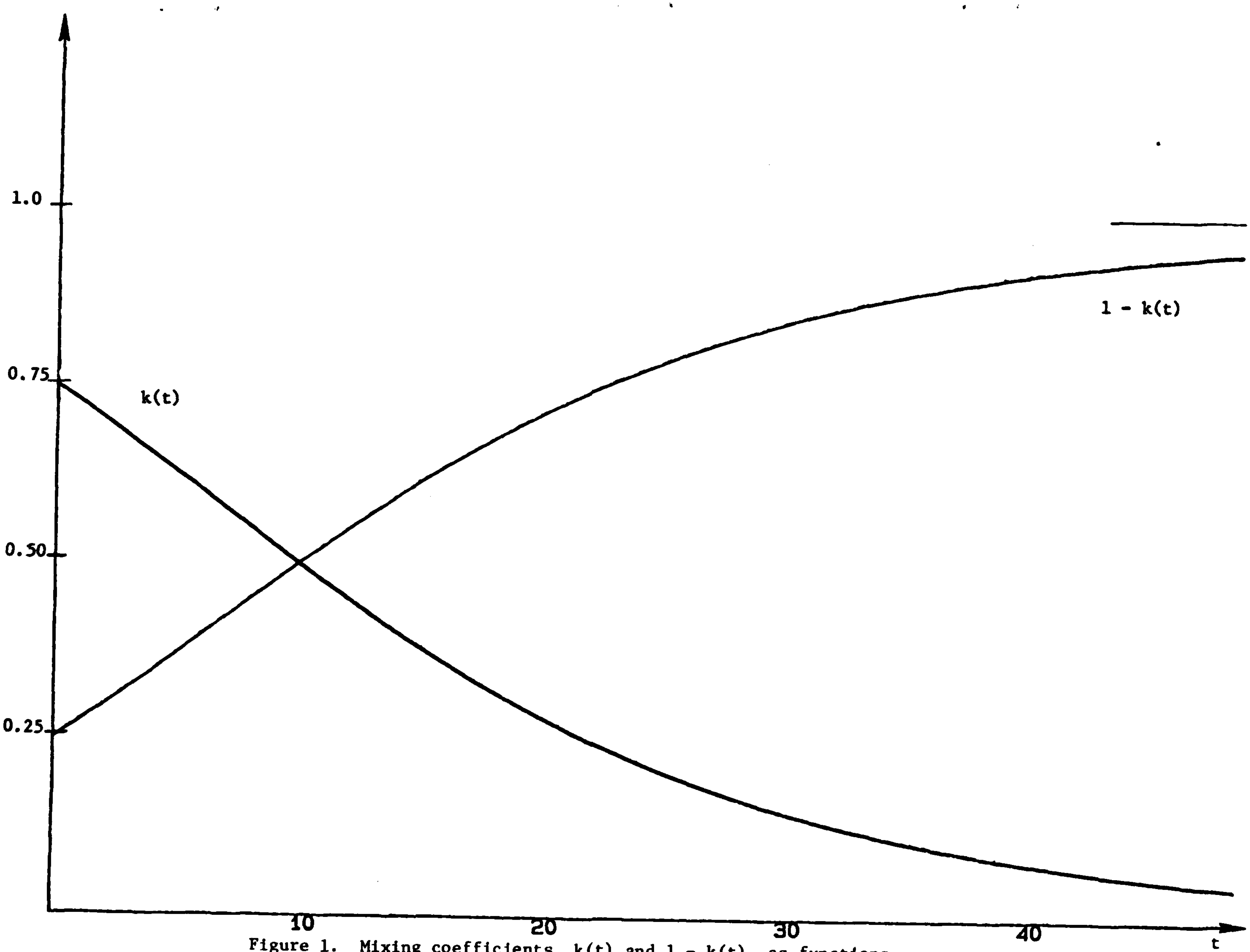

Figure 1. Mixing coefficients, $k(t)$ and $1 - k(t)$, as functions of non-detection interval, t .

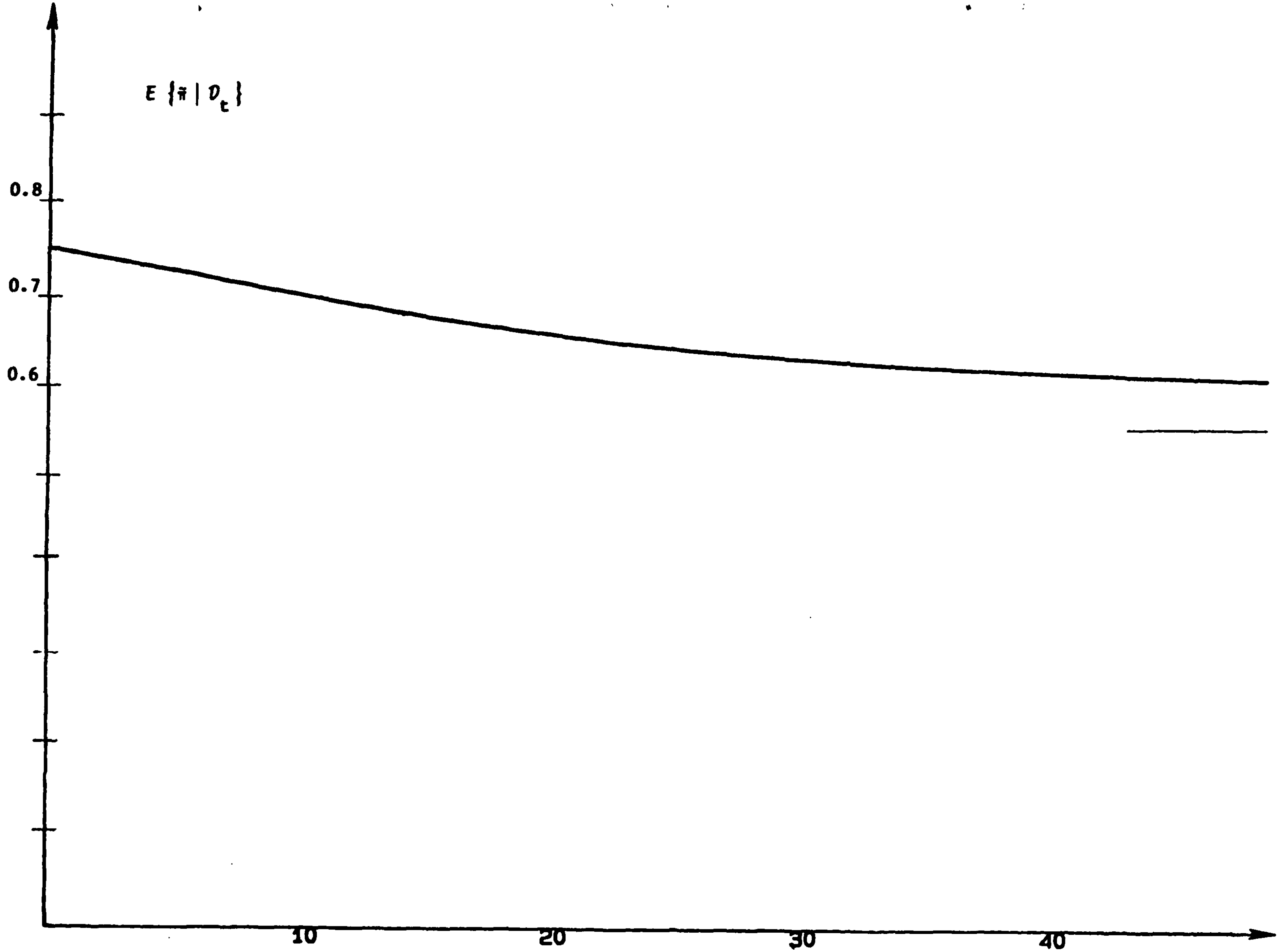

Figure 2. Posterior mean probability of object presence as a function of non-detection interval, t .

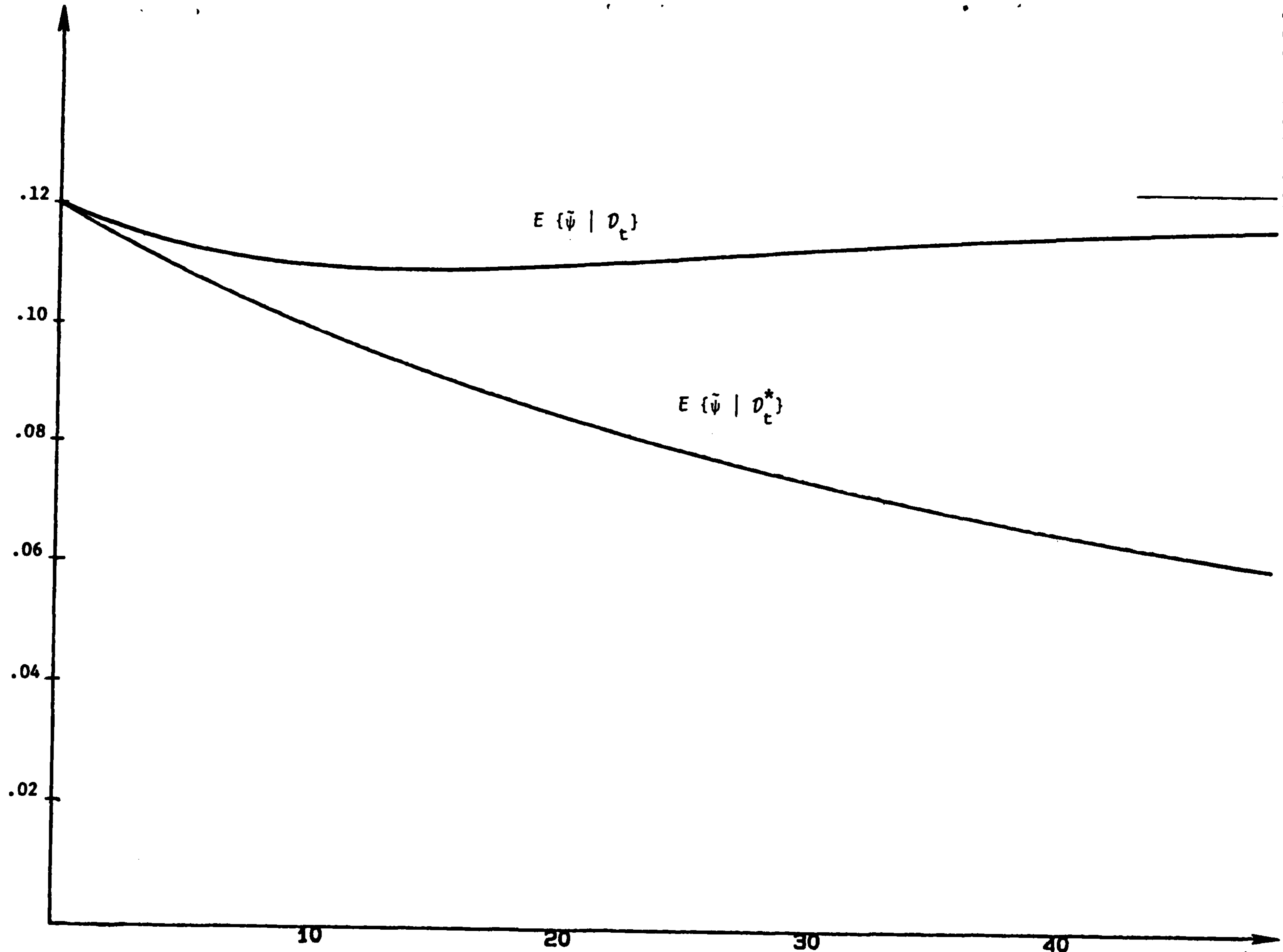

Figure 3. Posterior mean detection rate as a function of non-detection interval, t , for sample experiment ($\mathcal{D}_t^*$) and compound experiment ($\mathcal{D}_t$).

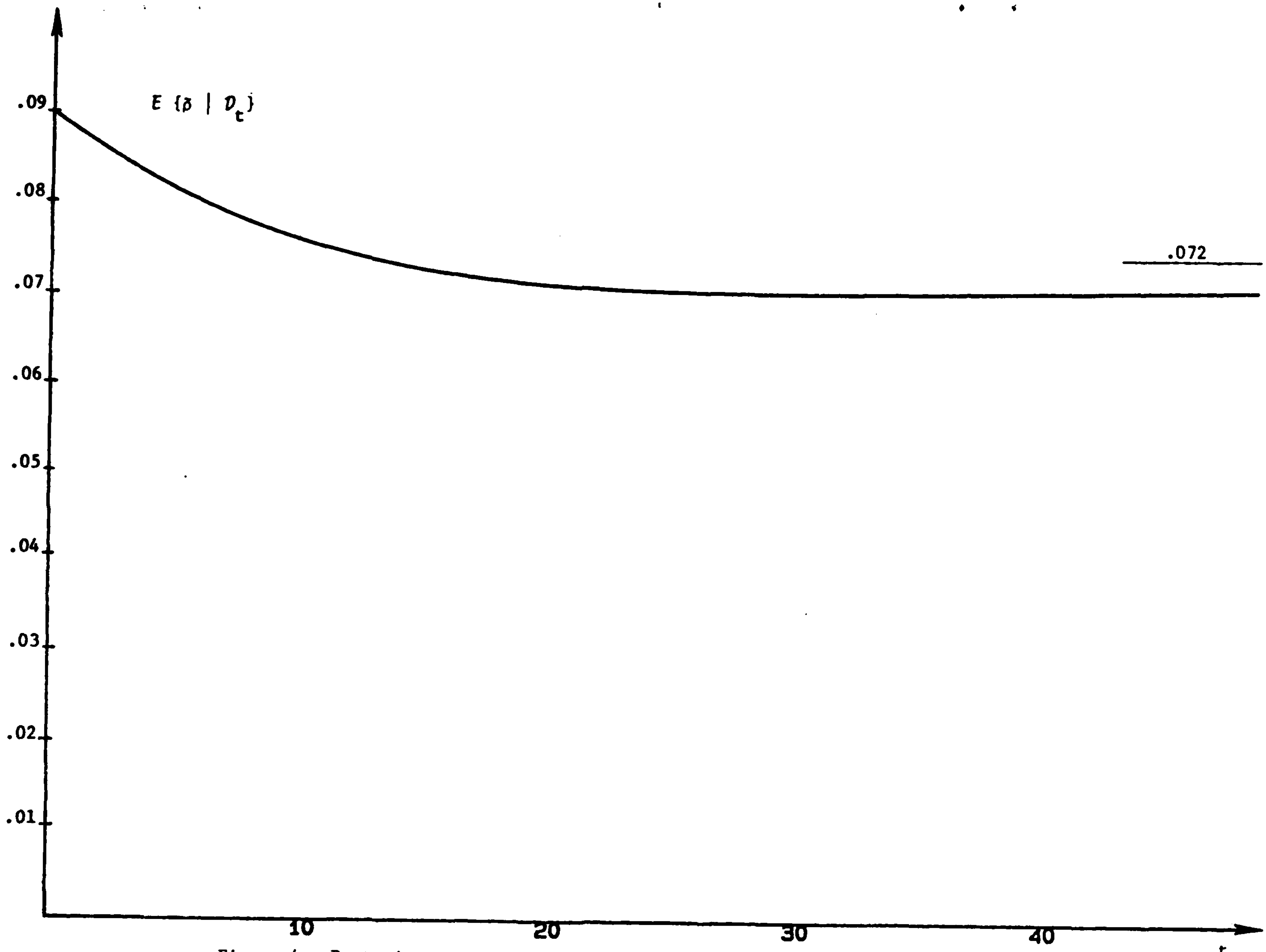

Figure 4. Posterior mean total discovery rate as a function of non-detection interval, t .

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